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Two-Port Analysis of SC Networks with Continuous Input Signals

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Switched-capacitor (SC) networks are comprised of capacitors interconnected by an array of periodically operated switches. Such networks are particularly attractive in light of the high circuit density possible with MOS circuit technology and hybrid integrated circuits using thin-film and silicon technology. Their implementation and analysis have received increasing attention over the last years, and economical designs find applications in communication and electronic equipment. Previous publications by the author and G. S. Moschytz have shown that SC networks can be analyzed as time-variant sampled-data networks by using nodal charge equations. This led to four-port equivalent circuits in the Z domain, thus modeling the SC network with a time-invariant network. The four-port equivalent circuit was reduced to a two-port, and a two-port transfer function for the entire SC network was derived. This previous work assumed that the network is fed by a staircase input function. In this paper, we show how the theory can be extended to continuous input functions. The results obtained are similar to those obtained by Liou and Kuo, but, since we use traditional two-port theory, the derivation is different. This derivation makes it simple and more intuitively accessible. The interpretation of the analytical expression leads to a bypass circuit that has the properties of a classical ring modulator added to the two-port equivalent circuit mentioned above. We present examples.

I. INTRODUCTION

Previous publications by the author and G. S. Moschytz showed that SC networks can be analyzed in the time domain by using nodal

charge equations with time-varying coefficients.[1,2,3] This led to a representation of SC networks as time-variant sampled-data networks, which can be modeled by a discrete, time-series, sampled-data system in cascade with sample-and-bold devices. Ultimately, it was shown that any SC network can be represented by a four-port equivalent circuit in the Z-domain.[2] Throughout this analysis, it was assumed that the SC network is fed by a staircase voltage function to be compatible with the inherent staircase nature of a voltage across any internal capacitor of the SC network. The charge transfer occurs at the end of the closing time of the switches; consequently, the magnitude of the voltage at every node of the network must be constant over the previous inactive interval of the switches. This leads to a staircase approximation of a compatible input function, as shown in Fig. 1. The closing times τ_1 and τ_2 of the two sets of switches are assumed to be different. One can define, however, a common period of $2\tau = \tau_1 + \tau_2$. With this common period of 2τ, it is possible to model the enitre SC network with a sampled-data system of a sampling rate $1/\tau$ followed by sample-and-hold circuits for restoring the physically realistic staircase output function. The following analysis forms the link to the previously explained ideas in Refs. 1 and 2. It is a prerequisite to the subsequent analysis with continuous input functions.

II. EXACT ANALYSIS WITH STAIRCASE INPUT

By continuing the ideas of a sampled-data system model of SC-networks as introduced in (1) and (2), the staircase input function shown in Fig. 1 can be expressed as

$$v_1^{st}(t) = \left\{ v_1(t) \cdot \sum_{\substack{n^e=0 \\ n^e=\text{even}}}^{\infty} \delta(t - n^e\tau) \right\} * f_2(t) + \left\{ v_1(t) \sum_{\substack{n^o=1 \\ n^o=\text{odd}}}^{\infty} \delta(t - \tau_1 - [n^o - 1]\tau) \right\} * f_1(t), \quad (1)$$

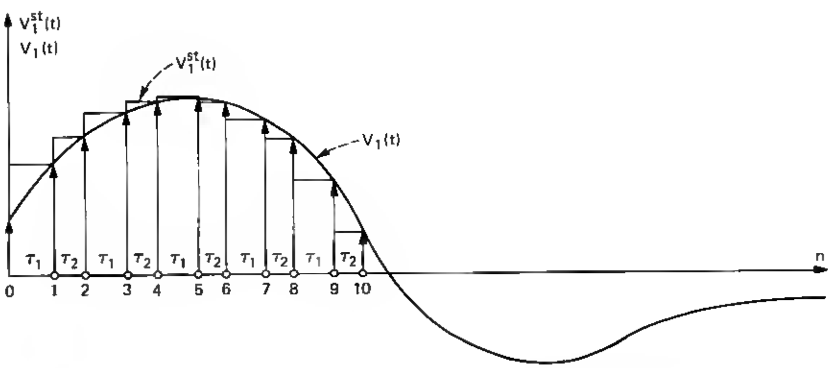

Fig. 1—Staircase approximation of input function to switched-capacitor network.

where (*) stands for convolution and $f_1(t)$ and $f_2(t)$ are the proper rectangular pulse functions of length τ_1 and τ_2, respectively, consistent with the ones used in section III B of Ref. 2. They convert a train of impulses into a staircase function. As can be noted, eq. (1) consists of two terms, a term with n^e = even sampling times and a term with n^o = odd sampling times. The second term with n^o = odd can be expressed with n^e = even by introducing $n^o = n^e + 1$. With a further substitution of variables,

$$t = t + \tau_1 - \tau - \Delta\tau \quad \text{and} \quad \Delta\tau = \frac{\tau_1 - \tau_2}{2},$$

eq. (1) can be rewritten as

$$v_1^{st}(t) = \left\{ v_1(t) \sum_{n^e=0}^{\infty} \delta(t - n^e\tau) \right\} * f_2(t)$$

$$+ \left\{ v_1(t + \tau_1 - \tau - \Delta\tau) \cdot \sum_{n^e=0}^{\infty} \delta(t - n^e\tau - \tau - \Delta\tau) \right\} * f_1(t). \quad (2)$$

Now in the term representing the odd samples, τ_1 can be considered a shift of the baseband signal $v_1(t)$ with respect to the sampling time $n^e \cdot \tau$, i.e., the instantaneous value of $v_1(n^e \cdot \tau)$. τ is a delay of the entire signal by one-half the previously defined period of $2\tau = \tau_1 + \tau_2$, and $\Delta\tau$ is a delay of the entire series of impulses after sampling. With a frequency variable of ω after sampling and a periodic variable of $z = e^{j\omega\tau}$ (corresponding to a Z-transform with a period of $\tau = 2\pi/\Omega_0$), eq. (2) can be expressed in the frequency domain:

$$V_1^{st}(\omega, \omega_0) = V_1(z^2) \cdot F_2(\omega) + V_1(z^2) z^{-1} e^{j\omega_0\tau_1} e^{-j\omega\Delta\tau} \cdot F_1(\omega). \quad (3)$$

$V_1(z^2)$ represents the input signal in the Z-domain sampled at a rate of ½ τ.

The frequency ω_0 in eq. (3) corresponds to the baseband frequency of the sampled signal $\{v_1(t) \leftrightarrow V_1(\omega_0)\}$ before sampling. It is related to the frequency variable ω by

$$\omega = \omega_0 + m\Omega_0/2 \qquad m = 0, 1, 2, 3, \cdots, \quad (4)$$

which later will be used in the spectral interpretation of eq. (3).

Note that $e^{j\omega_0\tau_1}$ is a phase shift for $v_1(t)$ before sampling. The frequency functions $F_1(\omega)$ and $F_2(\omega)$ are the functions corresponding to $f_1(t)$ and $f_2(t)$ in the time domain. They disperse the samples of $v_1(t)$ to rectangular pulses in the direction of the negative time axis and can be expressed as

$$F_1(\omega) = \frac{2 \sin \omega\tau_1/2}{\omega} e^{j\omega\tau_1/2} \quad (5a)$$

and

$$F_2(\omega) = \frac{2 \sin \omega\tau_2/2}{\omega} e^{j\omega\tau_2/2}. \tag{5b}$$

In pursuing the analysis of an sc network with a staircase input, only the discrete samples of the sampled-input function $v_1(t)$ are necessary to be considered as an input function, provided that the sc network has been modeled as a sampled-data system with a sampling rate of $1/\tau$ (or a period of 2τ and 50 percent duty cycle). This was explained in Refs. 1 and 2. It leads to a general block diagram of the entire sc network in the time or frequency domain as shown in Figs. 2 and 3.

The Z-transformed part of the sampled input function can be extracted from eq. (3) as

$$V_1^{st}(z) = V_1(z^2) + V_1(z^2)z^{-1}e^{j\omega_0\tau_1} = V_1(z^2)(1 + z^{-1}e^{j\omega_0\tau_1}). \tag{6}$$

The physical delay of $e^{-j\omega\Delta\tau}$ is applied to the odd samples at the output of the sampled-data system, which brings the sampling back to the nonequal sampling times τ_1 and τ_2 as shown in Fig. 3. $F_1(\omega)$ and $F_2(\omega)$ at the output convert the series of discrete samples into a staircase function, as it appears physically at the output of any sc network.

Finally, the even and odd samples have to be added into one consecutive string of pulses with alternating even and odd time slots.

With the four transfer functions, $H^{oo}(z)$, $H^{ee}(z)$, $H^{eo}(z)$ and $H^{oe}(z)$, as they appear in the model of the sc network,[2] the overall output signal can be expressed by inspecting Fig. 3 as

$$V_2^{st}(\omega) = V_2^e(z)F_2(\omega) + V_2^o(z)F_1(\omega) \tag{7}$$

$$V_2^e(z) = V_1(z^2)[H^{ee}(z) + H^{oe}(z)z^{-1}e^{j\omega_0\tau_1}] \tag{8a}$$

$$V_2^o(z) = V_1(z^2)[H^{eo}(z) + H^{oo}(z)z^{-1}e^{j\omega_0\tau_1}]e^{-j\omega\Delta\tau}, \tag{8b}$$

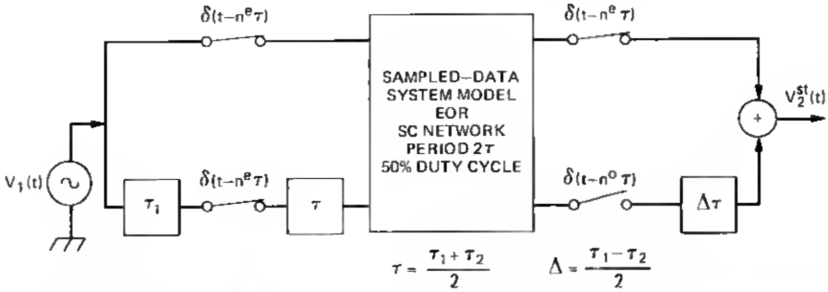

Fig. 2—Time domain equivalent circuit for switched-capacitor network with unequal switching times τ_1 and τ_2.

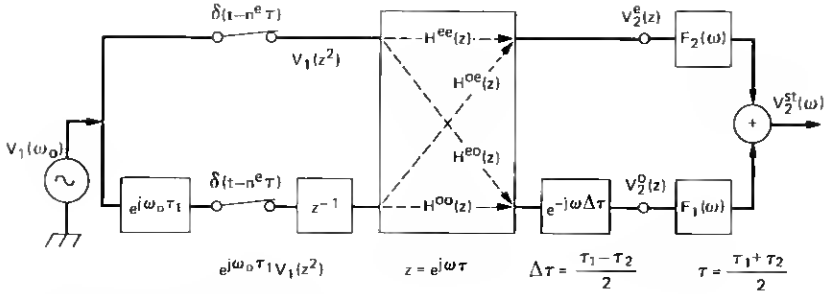

Fig. 3—Frequency-domain equivalent circuit for switched-capacitor network with unequal switching times τ_1 and τ_2.

and after substituting (8) into (7),

$$V_2^{st}(\omega) = V_1(z^2)\ \{[H^{ee}(z) + H^{oe}(z)z^{-1}e^{j\omega_0\tau_1}]\cdot F_2(\omega) + [H^{eo}(z) + H^{oo}(z)z^{-1}e^{j\omega_0\tau_1}]e^{-j\omega\Delta\tau}\cdot F_1(\omega)\}. \quad (9)$$

The four transfer functions $H(z)$ have to be determined from the four-port equivalent circuit in the Z-domain for any individual SC network under investigation. The derivation of the four-port equivalent circuit has been described in Refs. 1 and 2. In the latter part of this paper, examples will demonstrate the process of using it.

The significance of eq. (9) is that it describes the overall transfer characteristics of the entire SC network fed by a staircase input in one expression. This expression was derived in the frequency domain from a time-invariant model of the time-variant SC network. It might be pointed out that this time-invariant model has some similarity to the time-invariant models of periodically time-variant networks as described in Refs. 4 and 5.

In the following section, a correction term is introduced that extends the analysis from a staircase input to a continuous input function.

III. ANALYSIS FOR CONTINUOUS INPUT FUNCTIONS

By approximating the continuous input function $v_1(t)$ with a staircase function as indicated in Fig. 1, an error was made as shown in Fig. 4. This error can simply be calculated as

$$e_1(t) = v_1(t) - v_1^{st}(t). \quad (10)$$

It is important to notice that the error is always zero at sampling times, since the instantaneous values at sampling times have already been processed with the staircase input analysis. Consequently, what is left to be considered is the error function input signal between sampling times. From a physical inspection of any SC network, it can be determined that the network is time-invariant between two sam-

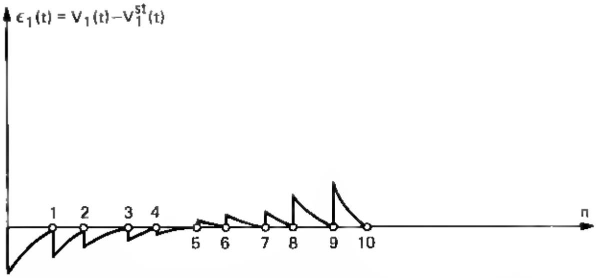

Fig. 4—Error function of the input of the switched-capacitor network.

pling instances. It behaves like a capacitive voltage divider that might have some active devices imbedded. The gain or loss of this capacitive voltage divider will change periodically when the switches alternate between positions for τ_1 and τ_2. Thus, any error voltage appearing within the closing times τ_1 or τ_2 will be attenuated (or amplified) according to the constant loss or gain of the network within these periods. The result is that a periodically modulated error signal will be superimposed on the output signal.

The modulating function can be determined by calculating the static *dc* gain of the sc network for the two different switching positions. A_1 is the gain for the switch position τ_1, and A_2 corresponds to τ_2 consequently, a switching function *S(t)* can be defined as shown in Fig. 5. It can be separated into a constant term A_0 and a function $A(t)$ as shown in Fig. 6, which alternates between a positive and negative value. This makes it possible to interpret the switching operation of an sc network as that of a conventional ring modulator with unequal switching times and a gain of $A_1 - A_0$, offset by a *dc* path of A_0. The resulting output error function $e_2(t)$ is the simple product of the input error $e_1(t)$ and the switching function

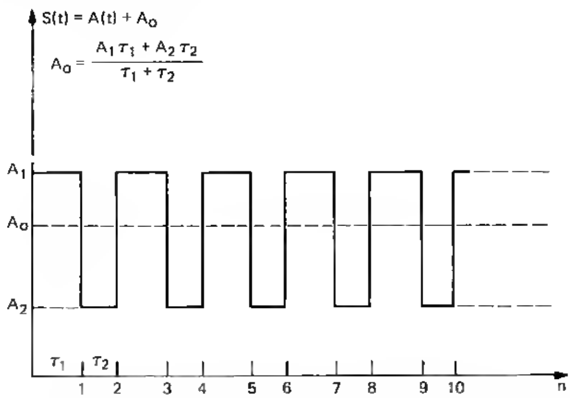

Fig. 5—Periodically changing gain of sc network between sampling times.

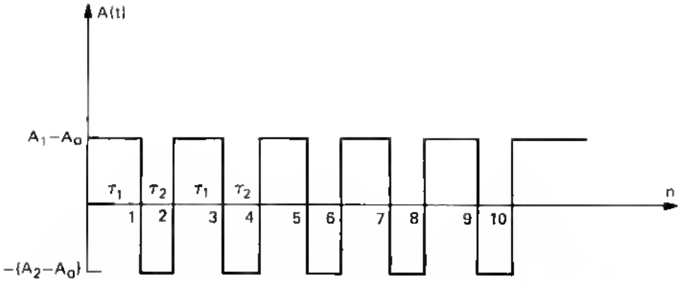

Fig. 6—Modulating function of sc network between sampling times.

$$e_2(t) = e_1(t) \cdot S(t), \tag{11}$$

or with eq. (10) and $S(t) = A_0 + A(t)$

$$e_2(t) = v_1(t) \cdot [A_0 + A(t)] - v_1^{st}(t) \cdot S(t). \tag{12}$$

Substituting into (12), the expression for $v_1^{st}(t)$ from eq. (2) and knowing that $S(t) = A_1$ for any time interval τ_1 and $S(t) = A_2$ for any time interval τ_2 results in

$$e_2(t) = v_1(t) \cdot A_0 + v_1(t) \cdot A(t) - A_2 \left\{ v_1(t) \cdot \sum_{n^e=0}^{\infty} \delta(t - n^e \tau) \right\} * f_2(t)$$
$$- A_1 \left\{ v_1(t + \tau_1 - \tau - \Delta\tau) \cdot \sum_{n^e=0}^{\infty} \delta(t - n^e \tau - \tau - \Delta\tau) \right\} * f_1(t). \tag{13}$$

With the previously introduced baseband frequency spectrum $V_1(\omega_0)$ before sampling and the Fourier transform of $A(t)$ as $\mathscr{F}\{A(t)\}$, and by using (3), eq. (13) can be expressed in the frequency domain:

$$E_2(\omega) = A_0 V_1(\omega_0) + V_1(\omega_0) * \mathscr{F}\{A(t)\} - A_2 V_1(z^2) \cdot F_2(\omega)$$
$$- A_1 V_1(z^2) z^{-1} e^{j\omega_0\tau_1} e^{-j\omega\Delta\tau} \cdot F_1(\omega), \tag{14}$$

where * is complex convolution.

Finally, the output in response to a continuous input function becomes the sum of eqs. (9) and (14).

$$V_2(\omega) = V_2^{st}(\omega) + E_2(\omega), \tag{15}$$

which yields

$$V_2(\omega) = A_0 V_1(\omega_0) + V_1(\omega_0) * \mathscr{F}\{A(t)\}$$
$$+ V_1(z^2)[H^{ee}(z) - A_2 + H^{oe}(z) z^{-1} e^{j\omega_0\tau_1}] F_2(\omega)$$
$$+ V_1(z^2)[H^{eo}(z) + (H^{oo}(z) - A_1) z^{-1} e^{j\omega_0\tau_1}] e^{-j\omega\Delta\tau} \cdot F_1(\omega). \tag{16}$$

The additional terms in eq. (16) can be included into the model by a slight modification of Fig. 3. This is shown in Fig. 7, which represents

the entire SC network with a continuous input signal. The terms A_1 and A_2 appear as frequency independent bypasses added into the stream of discrete samples. A_0 is a direct through path for the baseband frequency spectrum. The path through the modulator represents the term of the baseband signal multiplied by $A(t)$.

It may be appropriate to point out here that not all SC networks have a through path A_1 and A_2 during the closing times τ_1 and τ_2, respectively, of the switches. In many cases, A_1 and A_2 may be zero. Then Fig. 7 reduces to Fig. 3. Other cases may exist where A_1 (or A_2) only is zero. To better understand the physical meaning of eq. (16), it is necessary to interpret its spectral contents over the entire frequency range ω. This is the subject of the following section.

IV. SPECTRAL INTERPRETATION OF THE OUTPUT SIGNAL

Assuming the spectrum $V_1(\omega_0)$ of the input signal $v_1(t)$ is given as indicated in Fig. 8, the first term in eq. (16) will generate the same spectrum at the output of the system, scaled, however, by the constant factor A_0. It is shown on line 1 in Fig. 8. The spectrum is nonperiodic; consequently, $m = 0$ in eq. (4) and $\omega = \omega_0$ for this part of the output spectrum.

To evaluate the spectrum of the second term in eq. (16), it is necessary to know the spectrum of $A(t)$ as shown in Fig. 6. It is a

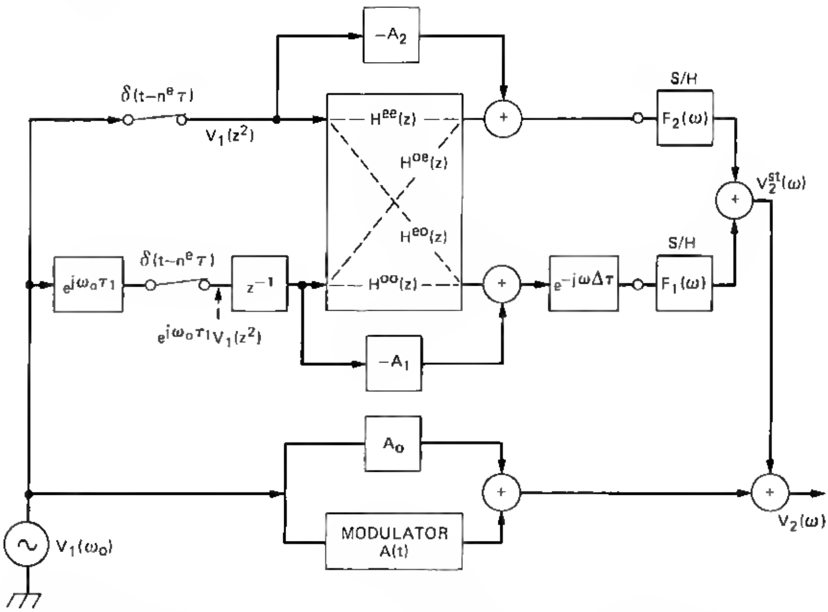

Fig. 7—Frequency domain model for SC network with continuous input and unequal switching times τ_1 and τ_2.

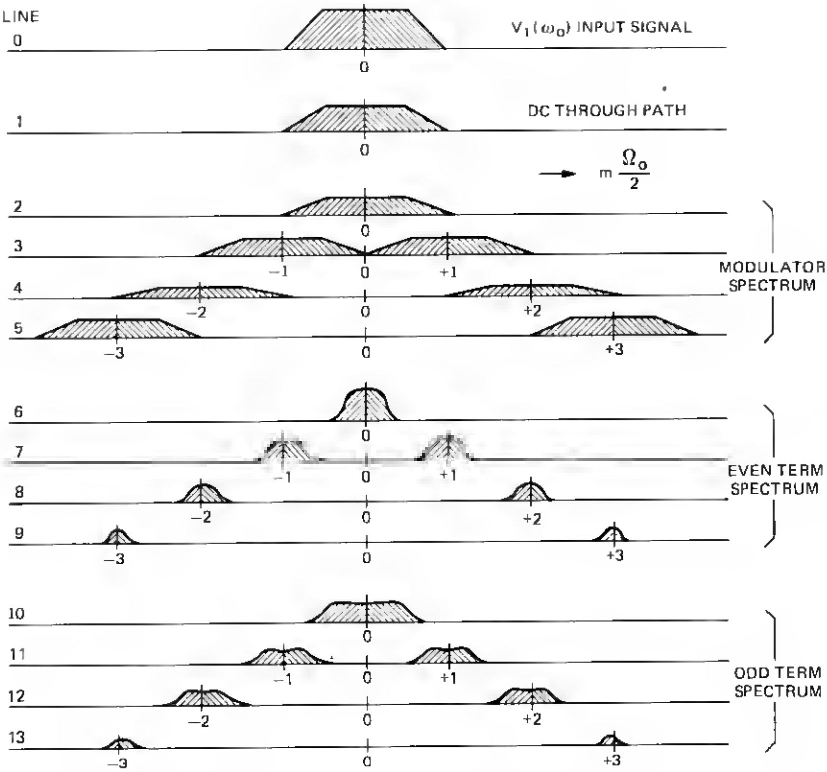

Fig. 8—Output spectrum distribution of SC network.

periodic signal; thus, its spectrum must be discrete. The function $A(t)$ can be expressed in the time domain as a series of impulses convolved with the function $f_1(t-\tau_1)$, which is the previously introduced function $f_1(t)$ delayed by τ_1. It disperses the impulses to pulses of finite width τ_1 in the positive time direction. With a correction term $(A_1 - A_2)\cdot \tau_1/(\tau_1+\tau_2)$, which assures a zero *dc* term, it follows:

$$A(t) = (A_1 - A_2)\left[\sum_{m=-\infty}^{m=+\infty} \delta(t - m2\tau)\right] * f_1(t-\tau_1) - (A_1 - A_2)\frac{\tau_1}{\tau_1+\tau_2}. \quad (17)$$

With the corresponding function of $F_1(\omega)$ as shown in eq. (5a), the spectrum of eq. (17) results in

$$\mathscr{F}\{A(t)\} = \frac{\Omega_0}{2}(A_1 - A_2)\left[\sum_{m=-\infty}^{m=+\infty} \delta(\omega - m\Omega_0/2)\right]\cdot \frac{2\sin\omega\tau_1/2}{\omega} e^{-j\omega\tau_1/2} - 2\pi(A_1 - A_2)\frac{\tau_1}{\tau_1+\tau_2}\delta(\omega), \quad (18)$$

which can also be written as

$$\mathscr{F}\{A(t)\} = 2(A_1 - A_2)\cdot \sum_{m=-\infty}^{m=+\infty} \frac{\sin m\,\dfrac{\Omega_0\tau_1}{4}}{m}\, e^{-jm\,\Omega_0\tau_1/4}\, \delta(\omega - m\Omega_0/2)$$

$$-2\,\pi(A_1 - A_2)\,\frac{\tau_1}{\tau_1 + \tau_2}\,\delta(\omega). \quad (19)$$

Now we can convolve eq. (19) with the baseband signal $V_1(\omega)$ to obtain the spectrum of the second term in eq. (16).

$$V_1(\omega) * \mathscr{F}\{A(t)\} = \frac{A_1 - A_2}{\pi}\cdot \sum_{m=-\infty}^{m=+\infty} \frac{\sin m\Omega_0\tau_1/4}{m}\, e^{-jm\Omega_0\tau_1/4}$$

$$\cdot\, V_1(\omega - m\Omega_0/2) - (A_1 - A_2)\,\frac{\tau_1}{\tau_1 + \tau_2}\, V_1(\omega). \quad (20)$$

As can be noticed, the spectrum $V_1(\omega) * \mathscr{F}\{At\}$ repeats periodically with $m\Omega_0/2$ over the entire frequency spectrum $-\infty < \omega < +\infty$ with a declining magnitude for higher m. This corresponds to the well-known performance of an ideal ring modulator with unequal switching times. By studying the declining magnitude of the periodic spectrum, it can be observed that the spectrum for even m is inherently smaller in magnitude. It will disappear completely for $\tau_1 = \tau_2 = \tau$, which is also a typical property of the ring modulator. The spectrum is indicated in Fig. 8, lines 2 to 5.

Finally, the spectrum of the last two terms of (16) can be explained by interpreting the spectrum of $V_1(z^2)$, which was the Z-transform of the sampled baseband signal $v_1(t)\cdot \sum_{m=-\infty}^{m=+\infty} \delta(t - m2\tau)$.

With the spectrum of the sampling function

$$\frac{\pi}{\tau} \sum_{m=-\infty}^{+\infty} \delta(\omega - m\Omega_0/2) \leftrightarrow \sum_{m=-\infty}^{+\infty} \delta(t - m2\tau), \quad (21)$$

the spectrum of $V_1(z^2)$ can be expressed by convolution

$$V_2(z^2) = V_1(\omega) * \frac{\pi}{\tau}\cdot \sum_{m=-\infty}^{+\infty} \delta(\omega - m\Omega_0/2) \quad z^2 = e^{-j\omega 2\tau}, \quad (22)$$

which is equal to

$$V_1(z^2) = \frac{1}{2\tau}\cdot \sum_{m=-\infty}^{+\infty} V_1(\omega - m\Omega_0/2) \quad (23)$$

Consequently, the last two terms in eq. (16) can be interpreted as the sum of the periodically spread-out baseband spectra shaped by a transfer characteristic depending on ω and m. With eq. (23), $z = e^{-j\omega\tau}$

and $\omega_0 = \omega - m\Omega_0/2$ substituted into eq. (16), we obtain:

$$\text{Term 3} = \frac{1}{2\tau} \sum V_1(\omega - m\Omega_0/2)$$

$$\cdot[H^{ee}(\omega) - A_2 + H^{oe}(\omega)e^{-j\omega\tau}e^{j(\omega - m\Omega_0/2)\tau_1}]\cdot F_2(\omega), \quad (24)$$

$$\text{Term 4} = \frac{1}{2\tau} \sum V_1(\omega - m\Omega_0/2)$$

$$\cdot[H^{eo}(\omega) + (H^{oo}(\omega) - A_1)e^{-j\omega\tau}e^{j(\omega - m\Omega_0/2)\tau_1}]e^{-j\omega\Delta\tau}\cdot F_1(\omega). \quad (25)$$

The spectra corresponding to eqs. (24) and (25) are indicated in Fig. 8 on lines 6–9 and 10–13. The functions $F_1(\omega)$ and $F_2(\omega)$ cause a decline of the magnitude of the spectra towards higher frequencies with a $(\sin x)/x$ roll-off.

The results of the above interpretations of the spectral parts can now be combined to a new expression for eq. (16) where the term $A_0 - (A_1 - A_2)\tau_1/(\tau_1 + \tau_2) = A_2$ is being used (see Fig. 5 for A_0):

$$V_2(\omega) = A_2 V_1(\omega) + \sum_{m=-\infty}^{m=+\infty} V_1\left(\omega - \frac{m\Omega_0}{2}\right)\left\{\frac{A_1 - A_2}{\pi}\,\frac{\sin m\Omega_0\tau_1/4}{m}\,e^{-jm\Omega_0\tau_1/4}\right.$$
$$+ [H^{ee}(\omega) - A_2 + H^{oe}(\omega)e^{-j\omega\tau}e^{j(\omega - m\Omega_0/2)\tau_1}]\,\frac{F_2(\omega)}{2\tau}$$
$$\left. + [H^{eo}(\omega) + \{H^{oo}(\omega) - A_1\}e^{-j\omega\tau}e^{j(\omega - m\Omega_0/2)\tau_1}]\,e^{-j\omega\Delta\tau}\cdot\frac{F_1(\omega)}{2\tau}\right\}. \quad (26)$$

Equation (26) can be used for calculating the power in any particular frequency range $\omega_1 < \omega < \omega_2$ of the entire spectrum by taking the integral of $|V_2(\omega)|^2$ over that range. Although this is a laborious process, it will be useful for the noise analysis of SC networks. For such noise analysis, $V_1(\omega)$ can be considered as the noise source at any given point of the network. The $H^{ik}(\omega)$ are the transfer characteristics of the network from the noise source to the output. Via superposition, the noise power from all sources can be accumulated at the output.

In closing, it might be pointed out that eq. (26) also can be expressed in terms of the baseband frequency ω_0 by substituting eq. (4). In addition, one can take advantage of the following observations:

$$z^2 = e^{j\omega 2\tau} = e^{j(\omega_0 + m\Omega_0/2)2\tau} = e^{j\omega_0 2\tau}e^{jm2\pi} = e^{j\omega_0 2\tau} \quad \text{for all m.}$$

This means that all even functions in z (or functions in z^2 only) like $H^{ee}(z)$ and $H^{oe}z^{-1}$ are periodic, and the following holds for all m:

$$H^{ee}(\omega_0 + m\Omega_0/2) = H^{ee}(\omega_0) \quad (27a)$$

and

$$H^{oe}(\omega_0 + m\Omega_0/2)e^{-j(\omega_0+m\Omega_0/2)2\tau} = H^{oe}(\omega_0)e^{-j\omega_0\tau}. \tag{27b}$$

Furthermore, the odd functions can always be considered as a function in z^2 multiplied by $z^{\pm 1}$. Consequently, with

$$z = e^{j\omega\tau} = e^{j(\omega_0+m\Omega_0/2)\tau} = e^{j\omega_0\tau}e^{jm\pi} = (-1)^m e^{j\omega_0\tau},$$

the odd functions in eq. (26) can be expressed as

$$H^{eo}(\omega_0 + m\Omega_0/2) = (-1)^m H^{eo}(\omega_0) \tag{28a}$$

and

$$H^{oo}(\omega_0 + m\Omega_0/2)e^{-j(\omega_0+m\Omega_0/2)\tau} = (-1)^m H^{oo}(\omega_0)e^{-j\omega_0\tau}. \tag{28b}$$

With this and resubstituting eq. (4) into (26), the final result is

$$\begin{aligned}
V_2(\omega_0 + m\Omega_0/2) = V_1(\omega_0)& \\
&\cdot\left\{A_2 + \sum_{m=-\infty}^{m=+\infty} \frac{A_1 - A_2}{\pi}\,\frac{\sin m\Omega_0\tau_1/4}{m}\,e^{-jm\Omega_0\tau_1/4}\right\} \\
&+ V_1(\omega_0) \\
&\cdot[H^{ee}(\omega_0) - A_2 + H^{oe}(\omega_0)e^{-j\omega_0\tau}e^{j\omega_0\tau_1}]\,\frac{1}{2\tau} \\
&\cdot \sum_{m=-\infty}^{m=+\infty} F_1(\omega_0 + m\Omega_0/2) \\
&+ V_1(\omega_0) \\
&\cdot[H^{eo}(\omega_0) + H^{oo}(\omega_0) - A_1 e^{-j\omega_0\tau}e^{j\omega_0\tau_1}]\,\frac{1}{2\tau} \\
&\cdot \sum_{m=-\infty}^{m=+\infty} (-1)^m e^{-j(\omega_0+m\Omega_0/2)\Delta\tau}F_2(\omega_0 + m\Omega_0/2).
\end{aligned} \tag{29}$$

Equations (26) and (29) describe any SC network in general terms and are equivalent to the slightly different expressions derived by M. L. Liou and Y. L. Kuo.[11] Both eqs. (26) and (29) are equally valid. Whichever is more convenient in a particular case can be used. It should be remembered that in eq. (29) each part of the spectrum is calculated relative to the frequencies $m\Omega_0/2$. This should be convenient for calculating just the spectrum relative to one particular frequency $m\Omega_0/2$ by knowing the baseband spectrum $V_1(\omega_0)$.

V. EXAMPLES

5.1 Double sample-and-hold circuit

As a simple first example, the analysis of a double sample-and-hold circuit shown in Fig. 9 is demonstrated. By applying the technique described in Ref. 1, a four-port equivalent circuit as shown in Fig. 10 can be established in the Z-domain. The "link two-ports" (LTPS) correspond to the storage capacitors C_1 and C_2 in Fig. 9. For a convenient further reduction of the circuit in Fig. 10, some properties of link two-ports may be recapitulated. As derived in Ref. 2, a convenient equivalent circuit for an LPT is the one shown in Fig. 11. If the LTP is driven by a voltage source, Thevenin's theorem yields an equivalent circuit as shown in Fig. 12. This is an important equivalent circuit, and it can be used advantageously to reduce complex configurations. Another important circuit element is an open-ended LTP whose input impedance frequently occurs in SC network equivalent circuits. Although derived in Ref. 2, its equivalent circuit is shown in Fig. 13 for the convenience of the reader. Finally, it is shown in Fig. 14 how an LTP reduces to two "non-linked" resistors by letting $z^{-1} = 0$. This special case is convenient for calculating the through-path coefficients A_1 and A_2 from the four-port equivalent circuit in the Z-domain.

After this short introduction, the circuit in Fig. 10 can be reduced to the one in Fig. 15 by using the equivalent circuit in Fig. 12.

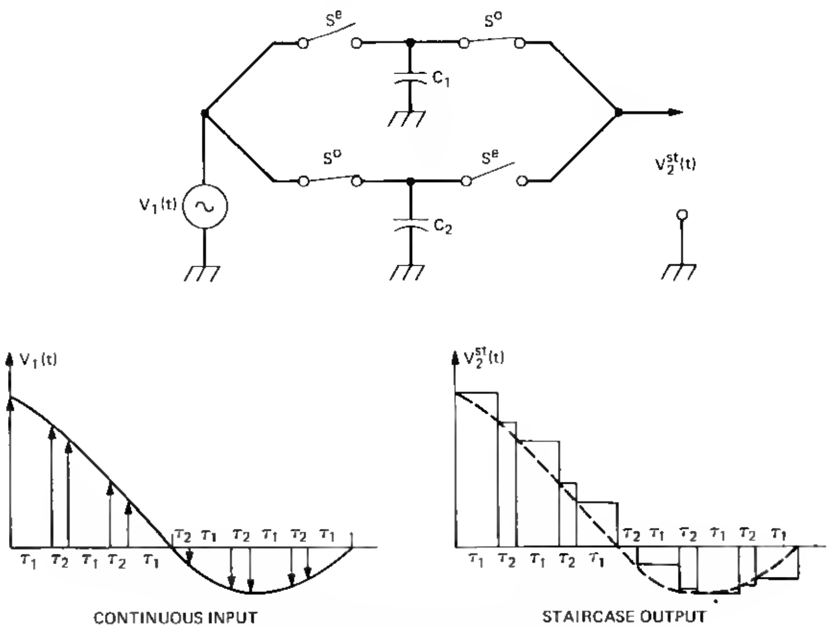

Fig. 9—Double sample-and-hold circuit.

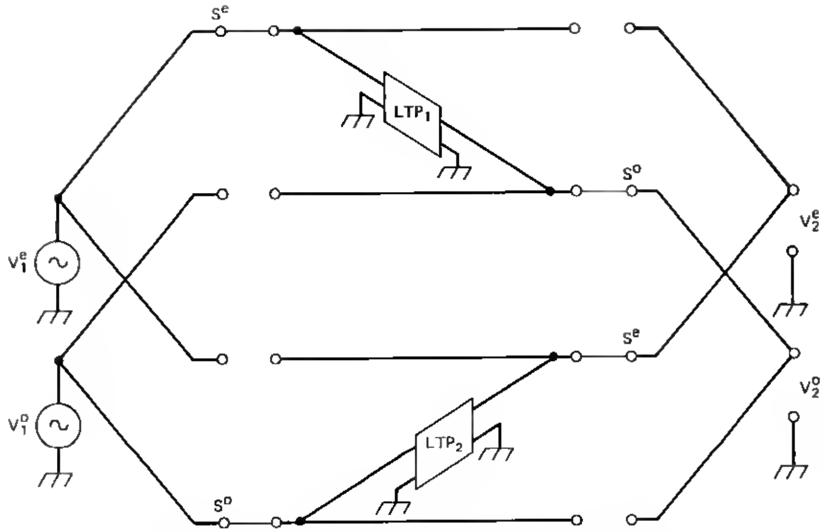

Fig. 10—Four-port equivalent circuit for double sample-and-hold circuit.

The four transfer functions can be established by inspecting Fig. 15.

$$H^{ee} = \frac{V_2^e}{V_1^e} = 0; \qquad H^{eo} = \frac{V_2^o}{V_1^e} = z^{-1} = e^{-j\omega\tau}$$

$$H^{oo} = \frac{V_2^o}{V_1^o} = 0; \qquad H^{oe} = \frac{V_2^e}{V_1^o} = z^{-1} = e^{-j\omega\tau}.$$

It also can be seen that the circuit does not have any static feed-through; consequently,

$$A_1 = 0 \quad \text{and} \quad A_2 = 0.$$

As a result, we obtain from eq. (26)

$$V_2(\omega) = \sum_m V_1(\omega - m\Omega_0/2)e^{-j\omega 2\tau}e^{j(\omega - m\Omega_0/2)\tau_1}\frac{F_2(\omega)}{2\tau}$$

$$+ \sum_m V_1(\omega - m\Omega_0/2)e^{-j\omega\tau}e^{-j\omega\Delta\tau}\frac{F_1(\omega)}{2\tau}. \qquad (30)$$

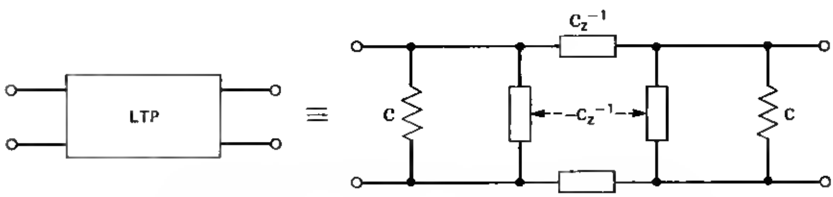

Fig. 11—Equivalent two port for link two ports.

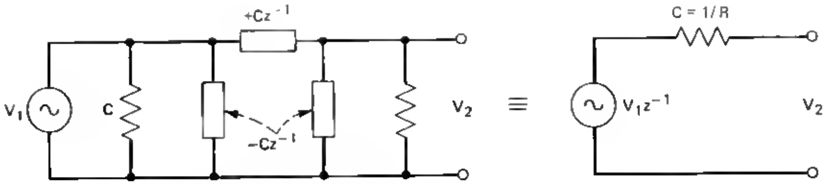

Fig. 12—Thevenin's equivalent for LTP driven with voltage source.

The output spectrum consists virtually of all terms for all m including $m = 0$. $F_1(\omega)$ and $F_2(\omega)$ introduce shaping towards higher frequencies.

A special case where $\tau_1 = \tau_2 = \tau$ is of interest. Under this condition, $F_1(\omega) = F_2(\omega)$ and (30) reduces to

$$V_2(\omega) = e^{-j\omega\tau}\frac{F_1(\omega)}{2\tau}\sum_m V_1(\omega - m\Omega_0/2)\{1 + (-1)^m\}$$

or, with eq. (5a), $\tau_1 = \tau$:

$$V_2(\omega) = \frac{1 - e^{-j\omega\tau_1}}{j\omega 2\tau}\sum_m V_1(\omega - m\Omega_0/2)\{1 + (-1)^m\}. \tag{31}$$

For odd m, the spectrum is zero, which indicates that the spectrum is related to a sampling period of τ. The double sample-and-hold circuit is of practical value, since it can be used to convert a continuous input signal into a staircase function, without any feed-through. The result is a spectrum that lacks all the extra terms shown in Fig. 8 on lines 2 to 5. The circuit can be used conveniently as driving circuit with resistive source impedances in the Z domain, as shown in Fig. 15. Any subsequent circuit can be designed exactly as a sampled-data system with a sample-and-hold circuit in tandem. Noise due to feed-through does not occur, regardless of potential feed-through paths in the subsequent SC network. It is not necessary to decouple the double sample-and-hold circuit from a following circuit via an active device.

5.2 First-order high-pass section

As a second example, a first-order high-pass section as shown in Fig. 16 will be analyzed. Again applying the technique described in Ref. 2,

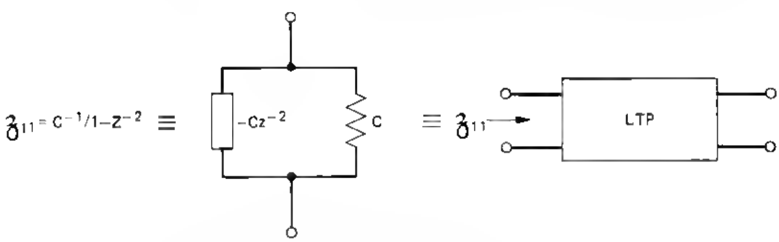

Fig. 13—Open circuit input impedance of LTP.

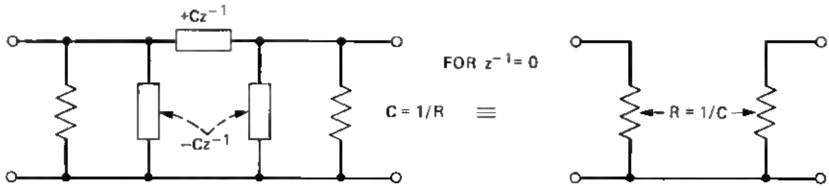

Fig. 14—LTP reduced to nonlinked two port for $Z^{-1} = 0$.

the equivalent four-port circuit as shown in Fig. 17 can be derived in the Z-domain. The four transfer functions can be derived as

(a) $$V_1^e = 0.$$

This results in $V_{22} = 0$, $V_{11} = 0$, $V_2^e = 0$. No current flows in the even path. Consequently,

$$H^{oo} = \frac{V_2^o}{V_1^o} = 1 \tag{32a}$$

$$H^{oe} = \frac{V_2^e}{V_1^o} = 0 \tag{32b}$$

(b) $$V_1^o = 0.$$

No current flows in the even path, which corresponds to an open-ended LTP in the even path. Consequently,

$$H^{ee} = \frac{V_2^e}{V_1^e} = \frac{R_2}{R_2 + Z_{11}} = \frac{C_1}{C_1 + C_2/(1 - z^{-2})}$$

where Z_{11} was substituted with the expression in Fig. 13.

Applying the equivalent circuit in Fig. 12 yields $V_{22} = V_{11}z^{-1}$. V_{11} can be calculated from V_1^e via voltage division

$$V_{11} = V_1^e \frac{Z_{11}}{R_2 + Z_{11}} = V_1^e \frac{C_2}{C_1(1 - z^{-2}) + C_2},$$

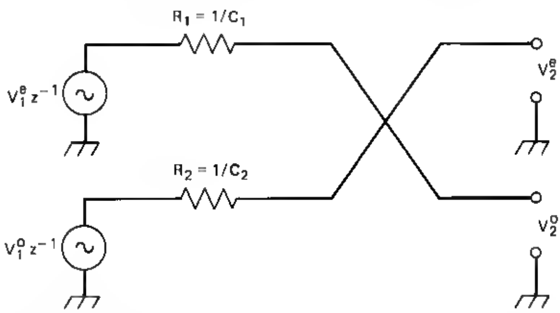

Fig. 15—Reduced four-port equivalent circuit for double sample-and-hold circuit.

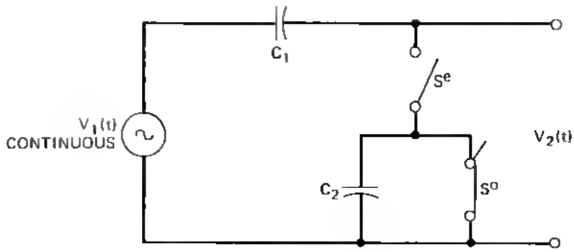

Fig. 16—First-order high-pass section.

which yields

$$H^{eo} = \frac{V_2^o}{V_1^e} = \frac{-V_{22}}{V_1^e} = \frac{-V_{11}z^{-1}}{V_1^e} = \frac{-z^{-1}C_2}{C_1(1 - z^{-2}) + C_2}.$$

With $\beta = C_2/(C_1 + C_2)$ H^{ee} and H^{eo} can be rewritten more conveniently:

$$H^{ee} = \frac{(1-\beta)(z^2 - 1)}{z^2 - 1 + \beta} \tag{33a}$$

$$H^{eo} = \frac{-\beta z}{z^2 - 1 + \beta}. \tag{33b}$$

Finally, the coefficients A_1 and A_2 of the through path can be calculated by substituting $z^{-1} = 0$ into the LTP, as was explained in Fig. 14. This

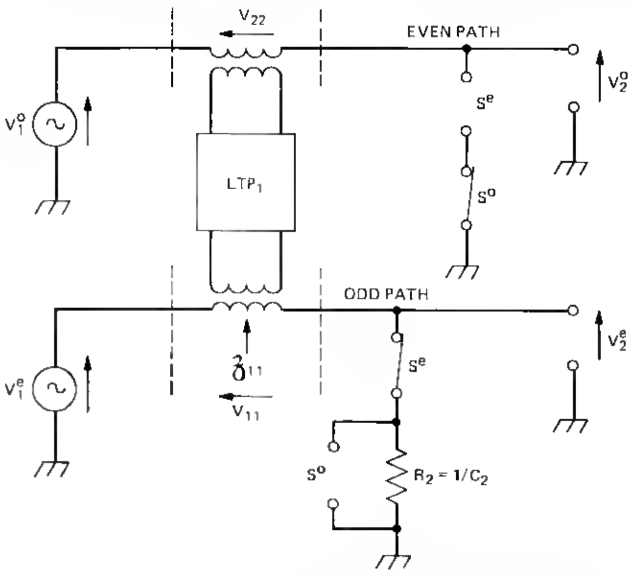

Fig. 17—Four-port equivalent circuit for high-pass section.

eliminates all memory and separates the even and odd paths completely. The static value of the storage capacitor C_1, which corresponds to a resistor, remains in each path. This is indicated in Fig. 18, from which A_1 and A_2 can be derived by inspection [A_1 corresponds to odd samples and A_2 to even samples—see eq. (13)].

$$A_1 = H^{oo}(0) = \frac{V_2^o}{V_{1_{z=\infty}}^o} = 1 \tag{34a}$$

$$A_2 = H^{ee}(0) = \frac{V_2^e}{V_{1_{z=\infty}}^e} = \frac{R_2}{R_1 + R_2} = 1 - \beta. \tag{34b}$$

Substituting eqs. (32), (33), and (34) into eq. (26) results in the following expression for the entire output spectrum:

$$\begin{aligned} V_2(\omega) = {} & (1-\beta)V_1(\omega) \\ & + \frac{\beta}{\pi}\sum_m V_1(\omega - m\Omega_0/2)\frac{\sin m\omega_0\tau_1/4}{m}\,\mathrm{e}^{-jm\Omega_0\tau_1/4} \\ & + \sum_m V_1(\omega - m\Omega_0/2)\frac{-\beta(1-\beta)}{\mathrm{e}^{j\omega 2\tau} - 1 + \beta}\,\frac{F_2(\omega)}{2\tau} \\ & + \sum_m V_1(\omega - m\Omega_0/2)\frac{-\beta e^{j\omega\tau}e^{-j\omega\Delta\tau}}{\mathrm{e}^{j\omega 2\tau} - 1 + \beta}\,\frac{F_1(\omega)}{2\tau}. \end{aligned} \tag{35}$$

As we can see, virtually all possible parts in the spectrum are present. Notice that parts of the direct terms $V_1(\omega)$ will be cancelled

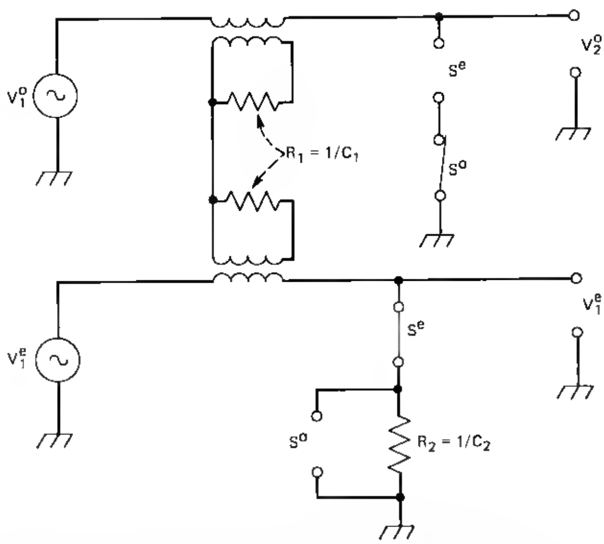

Fig. 18—Four-port equivalent circuit for high-pass section for $Z^{-1} = 0$.

by the baseband content of the other terms. The example, although a relatively simple circuit, shows how complex the analysis can become when the input signal is not a staircase function but rather a continuous function. This particular result in eq. (35) is an example of where a transformation into the form shown in eq. (29) might be advantageous. It allows an easy calculation of every spectral component around $m\Omega_0/2$ separately. The same example was treated by Tsividis[9,10]. After further reduction, eq. (35) leads to the expression obtained by Liou and Kuo[11,12].

VI. CONCLUSIONS

We have shown how the two-port analysis of SC networks, which was based on staircase inputs, can be extended to continuous input functions. This extension still allows us to model the SC network as a four-port representing a sampled-data system in tandem with sample-and-hold circuits. After a slight modification of this model, continuous input signals can be treated. For convenience or for practical reasons, most SC network designs will be based on the staircase approximation of the input signal. For any noise analysis or the introduction of nonlinear terms, an analysis with a continuous input signal is necessary. The method shown demonstrates how an SC network designed for a staircase input can be post-analyzed with respect to its response to a continuous input signal. As explained, the numerical analysis can be achieved with computer programs based on mesh or nodal analysis principles and a post-processor for the extension to continuous input signals.

VII. ACKNOWLEDGMENT

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REFERENCES

1. C. F. Kurth and G. S. Moschytz, "Nodal Analysis of Switched-Capacitor Networks," IEEE Trans. CAS, *26*, No. 2 (Feb. 1979), pp. 93–105.
2. C. F. Kurth and G. S. Moschytz, "Two-Port Analysis of Switched-Capacitor Networks Using Four-Port Equivalent Circuits," IEEE Trans. CAS, *26*, No. 3 (March 1979), pp. 166–180.
3. K. R. Laker, "Equivalent Circuits for the Analysis and Synthesis of Switched Capacitor Networks," B.S.T.J., *58*, No. 3 (March 1979), pp. 729–769.
4. C. A. Desoer, "Steady-State Transmission Through a Network Containing a Single Time-Variant Element," IRE Trans. Circuit Theory (Sept. 1959), pp. 249–252.
5. C. F. Kurth, "Steady-State Analysis of Sinusoidal Time-Variant Networks Applied

to Equivalent Circuits for Transmission Networks," IEEE Trans. CAS, *24*, No. 11 (Nov. 1977).
6. A. Papoulis, *The Fourier Integral and its Applications*, New York: McGraw-Hill, 1962.
7. E. I. Jury, *Theory and Application of the Z-Transform Method*, New York: John Wiley, 1964; republished by E. Kreiger Pub. Co., 1973.
8. G. Doetsch, *Guide to the Applications of the Laplace and Z-Transforms*, London: Van Nostrand Reinhold, 1971.
9. Y. Tsividis, "Analytical and Experimental Evaluation of a Switched-Capacitor Filter and Remarks on the Resistor/Switched-Capacitor Correspondence," IEEE Trans. CAS, *26*, No. 2 (Feb. 1979).
10. Y. Tsividis, "Analysis of Switched-Capacitor Networks," IEEE Trans. CAS, (Nov. 1979).
11. M. L. Liou, Y. L. Kuo, "Exact Analysis of Switched-Capacitor Circuits with Arbitrary Inputs," IEEE Trans. CAS, *26*, No. 4 (April 1979), pp. 213–223.
12. Y. L. Kuo, M. L. Liou, J. W. Kasinskas, "An Equivalent Circuit Approach to the Computer-Aided Analysis of Switched Capacitor Circuits," IEEE Trans. CAS, *26*, No. 9 (Sept. 1979), pp. 708–714.